Bounding the Prime Factors of Odd Perfect Numbers

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Abstract

Odd perfect numbers have long been studied by mathematicians; their nonexistance is the longest-standing unsolved conjecture in number theory and, indeed, all of mathematics. Bounds for each of the distinct prime factors of an odd perfect number are derived, drawing heavily on existing works. This is done in cases based on the total number of distinct prime factors.

Further results in the field are studied and examples of various techniques are given, including an extended sigma chain. The paper concludes with explicit double-sided bounds for each of the factors of odd perfect numbers for each of 8 through 19 distince prime factors.

1 Motivation

The study of perfect numbers has motivated the development of much of number theory. The ancients had calculated the first four perfect numbers, and had formulated several theories about them. Euclid implicitly assumed that all perfect numbers were even, and it was not until the time of Descartes that the possibility of odd perfect numbers was considered.

No odd perfect numbers have been found, but many theorems restricting their factors have been discovered. Perhaps an odd perfect number exists; perhaps none exist and a proof will be written someday. It is even possible that no odd perfect numbers exist but no proof of this fact is possible within the accepted system of axoims.

2 Terminology

The sum of divisors function σ is defined by

$$\sigma(p_1^{a_1}p_2^{a_2}\cdots p_n^{a_n}) = \sigma(m) = \sum_{\substack{d|m \\ d\geqslant 1}} d = \prod_{i=1}^{n}(1+p_i+p_i^2+\cdots+p_i^{a_i}) = \prod_{i=1}^{n}\frac{p_i^{a_i+1}-1}{p_i-1}$$

N is said to be perfect if $\sigma(N) = 2N$.

Let N denote an odd perfect number with factorization $N = p_1^{a_1}p_2^{a_2}\dots p_t^{a_t}$ with $p_1,\dots,p_t$ distinct primes and $p_1 > \dots > p_t$.

The abundancy function $\sigma_{\text{-}1}$ is defined by

$$\sigma_{\text{-}1}(m) = \sigma(m)/m$$

Write $q^a \parallel M$ if $q^a \mid M$ and $q^{a+1} \nmid M$, where q is prime.

3 Introduction

Euler was first to study seriously the possibility of odd perfect numbers. He realized that since $N \equiv 2 \pmod 4$ that of the prime factors of N, all but one would be raised to an even power and the remaining 'special' prime has the form p^a, where $p \equiv 1 \equiv a \pmod 4$.

Proof: It follows from the definition of σ that $2 \mid \sigma(p^{2k+1})$; in particular, $2 \parallel \sigma(p^{4k+1})$ and $4 \mid \sigma(p^{4k+3})$. Clearly, none of the prime factors of N can be raised to a power equivalent to 3 modulo 4, and only one can be raised to the power of an exponent equivalent to 1 modulo 4. This establishes that all but one exponents are even and the special prime's exponent is $\equiv 1 \pmod 4$.

If the special prime was $\equiv 3 \pmod 4$ then $N \equiv 3 \pmod 4$, since the other primes are odds raised to an even power and are thus $\equiv 1 \pmod 4$. But this is a contradiction, since (as in Touchard [27]) $\frac{N^4(N-1)}{12}$ is an integer, so if $N \equiv 3 \pmod 4$ then $2 \mid N^4$ (which is absurd; N is by definition odd).

Arguably the most important tool for studying perfect numbers is the abundancy function. It has several properties that make it so important. First (recalling the definition of abundancy), since the divisor function is multiplicitive, so is the abundancy function.

Second, for a given prime power p^a

$$\lim_{a \to \infty} \sigma_{\text{-}1}(p^a) = \frac{p}{p-1}$$

Also, note that $\sigma_{\text{-}1}(p) = 1 + \frac{1}{p} = \frac{p+1}{p}$. The value of the abundancy function for a prime power p^a is therefore in the range

$$\frac{p+1}{p} \leqslant \sigma_{\text{-}1}(p^a) < \frac{p}{p-1} \tag{1}$$

Further, $\sigma_{\text{-}1}(p^a) > \sigma_{\text{-}1}(p^b)$ when $a > b$.

Rewriting the conditions for the perfect number,

$$\begin{aligned} \sigma(N) &= 2N \\ \sigma(N)/N &= 2 \\ \sigma_{\text{-}1}(p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}) &= 2 \\ \sigma_{\text{-}1}(p_1^{a_1})\, \sigma_{\text{-}1}(p_2^{a_2}) \cdots \sigma_{\text{-}1}(p_t^{a_t}) &= 2 \end{aligned}$$

4 Finding the Bounds

These results allow some restrictions on the values of the prime factors of N. For example, $3{\cdot}5{\cdot}7 \nmid N$. Of these three primes, only $5 \equiv 1 \pmod 4$, so the exponents on 3 and 7 must be even. Since increasing the exponent on a prime increases the abundancy, we have $2 \geqslant \sigma_{\text{-}1}(3^2 \cdot 5 \cdot 7^2) = \frac{1+3+9}{9}\frac{1+5}{5}\frac{1+7+49}{49} = \frac{494}{245} = 2.0163265\ldots > 2$, a contradiction. As a result, the third smallest prime $p_{t-2} \geqslant 11$.

To derive further bounds, though, it is advisable to use some of the many published results regarding odd perfect numbers. Appendix A contains some of the most important results.

Two have direct relevance to the subject of this paper, and as such will not be improved upon in this paper. Jenkins [4] proved that $p_1 > 10^7$. Iannucci [7][9] proved that $p_2 > 10^4$ and $p_3 > 10^2$.

Hagis [14] and Kishore [15] independently proved that if $3 \nmid N$ that the number of distinct prime factors t of N is at least 11. Therefore, for $8 \leqslant t \leqslant 10$, we have $p_t = 3$.[1]

[1] Hagis [18] and Chein [19] proved that all odd perfect numbers have at least 8 factors.

A result due to Perisastri[2] [26] appears to be useful: $p_t < \frac{2}{3}t + 2$. Results like the above reduce its practicality. In fact, a paper by Norton [25] just two years later supercedes Perisastri's formula, except that Norton's method is slightly more computationally intensive.

Of course, with the application of Iannucci's results, we can modestly improve on his bounds with an otherwise straightforward application of the abundancy function. The result is that $t \geqslant 17$ if $p_t \geqslant 7$ and $t \geqslant 29$ if $p_t \geqslant 11$.

A useful result here is an earlier work of Kishore [16], in which he proves that

$$p_{t-i} < 2^{2^i}(t-i) \forall 1 \leqslant i \leqslant 5 \tag{2}$$

This allows us to explicitly reduce the bounds for the lowest 6 prime factors for an odd perfect number with a given number of distinct prime factors. For example, a perfect number with 8 distinct factors has $p_8 \leqslant 7$ (the smallest prime strictly less than $\frac{16}{3} + 2$), $p_7 \leqslant 23$ (the smallest prime less than $2^{2^1} \cdot 7$), $p_6 \leqslant 89$, $p_5 < 2^8 \cdot 5$, $p_4 < 2^{18}$, and $p_3 < 2^{32} \cdot 3$.

Using the abundancy function, the bounds for p_7 and p_6 can be reduced. If $p_7 \geqslant 13$ then

$$\sigma_{\text{-}1}(N) \leqslant \sigma_{\text{-}1}(3^a 13^b 17^c 19^d 23^e 101^f 10\,007^g 10\,000\,019^h) < \frac{3}{2}\frac{13}{12}\frac{17}{16}\frac{19}{18}\frac{23}{22}\frac{101}{100}\frac{10\,007}{10\,006}\frac{10\,000\,019}{10\,000\,018} < 2$$

so $p_7 \leqslant 11$. Similarly, if $p_8 = 3, p_7 = 5, p_6 \geqslant 53$ then $\sigma_{\text{-}1}(N) < 1.997$, so $p_6 \leqslant 47$.

A major problem with the abundancy function is that it is incapable of determining upper limits on primes beyond the smallest three (except in restricted cases).

This is because the first three primes could be 3, 5, and 11, in which case the sigma bounds allow an arbitrary number of additional prime factors, but require no more.

In a recent article published in the electronic journal INTEGERS, Nielsen [5] extends earlier results by showing that $N < 2^{4^t}$. Obviously $p_i < N$, so this is an implicit upper bound on p_1. If p_1 is the special prime with an exponent of 1 and the other p_i are small, we can say little else about p_1. We can, however, restrict the other primes more closely.

Since only one of p_1, p_2 can be the special prime, the exponent on at least one is even. Thus $p_1 p_2^2 < N < 2^{4^t}$, so $p_2 < \sqrt[3]{2^{4^t}} = 2^{4^t/3}$. Similarly, for $1 \leqslant i \leqslant t$:

$$p_i < 2^{4^t/(2i-1)} \tag{3}$$

This range can be restricted further by considering the other prime factors. There are at least 75 total primes (not distinct) according to Hare [1], so we make take the other primes as small as possible and raise the lowest prime to the appropriate power (and the others to the 2nd power). This can reduce the bound by perhaps a million, depending on t.

This gives explicit formulations (2) for the upper bounds on the smallest 6 prime factors, which are supplemented by sigma conditions for the lowest 3. The higher prime factors are similarly restricted (3), though not as tightly.

Finding lower bounds for the highest prime factors is a common topic for papers in the field; Jenkins [4] and Iannucci [7][9] have the most recent work to my knowledge. Both Iannucci and Jenkins use proofs based on the divisibility of cyclotomic polynomials $F_p(x)$.

Going beyond the above results, though, is not easy. The method used in most proofs is that of sigma chains (explained in more detail in Appendix B). Consider an odd perfect number N with a component 5^{4k+1}. We know that

$$\sigma(5) \mid \sigma(5^{4k+1})$$

for all natural k, so we know that $\sigma(5) \mid 2N$. $\sigma(5) = 6$, so we know that $3 \mid N$.

[2] This result may have been first proved by Grün; I have conflicting references. Regardless, I find myself disinclined to research the matter myself, as it is only tangentially related to the paper itself and would require finding copies of an old mathematical proof written in German.

In this manner we can construct a chain based on the prime factors of $\sigma(q^a)$. A particular chain is terminated when either basic principles are violated (e.g. the abundancy is already greater than 2), or when the conditions one set out to prove have been violated.

Sigma chains might be used to determine if it is possible for an odd perfect number to have a component less than 10^{30} (see Appendix B as an example of this method).

In fact, few modern proofs about odd perfect numbers would be possible without sigma chains. The proofs of Hare [1] [2], Iannucci [7] [9], Jenkins [4], Nielsen [5], and many others would have been impossible otherwise. Of course, each of these sigma proofs is different, having been optimized for different situations. Sometimes bounds are checked in parts, with each individual part contributing a fact toward the proofs of the other parts.

There are other methods used besides sigma chains, though there is no good generalization since these other methods are generally specific to their application. Touchard [27] has an interesting proof, based on Balth van der Pol's equation

$$\alpha(t) = 1 - \sum_{k=1}^{\infty} \sigma(k) e^{-kt}$$

Using this function in differential form and substituting, he obtains

$$n^3(n-1)\sigma(n) - 48nS_2 + 72S_3 = 0$$

where $S_p \equiv S_p(n-1) = \sum_{k=1}^{n-1} k^p \sigma(k)\sigma(n-k)$.

In the particular case where $n = N$ is an odd perfect number, this is of course the same as

$$n^4(n-1) = 24nS_2 - 36S_3$$

where $24nS_2 - 36S_3$ is an integer. This proves Touchard's theorem that N is 1, 9, 13, or 25 modulo 36.

While this does not of itself give bounds on the primes, it is useful in proving further bounds.

A Facts About Odd Perfect Numbers

$N = p_1^{a_1} p_2^{a_2} \ldots p_t^{a_t}$ with $p_1, \ldots, p_t$ distinct primes and $p_1 > \ldots > p_t$.

- $p_1 \geqslant 10\ 000\ 019$ from Jenkins [4], improving on Hagis and Cohen [10]
- $p_2 \geqslant 10\ 007$ from Iannucci [9], improving on Pomerance [20]
- $p_3 \geqslant 101$ from Iannucci [7]
- $p_{t-i} < 2^{2^i}(t-i)$ for $1 \leqslant i \leqslant 5$ from Kishore [16]
- $p_t < \frac{2}{3}t + 2$ from Perisastri [26]
- $p_k^{a_k} > 10^{20}$ for some k from Cohen [13], improving on Muskat [24]
- $N > 10^{300}$ from Brent, et. al. [11], improving on Brent and Cohen [12]
- $N \equiv 1, 9, 13, 25 \pmod{36}$ from Touchard [27]
- $N < 2^{4^t}$ from Nielsen [5], improving on Cook [8]
- $a_1 + a_2 + \ldots + a_t = \Omega(N) \geqslant 75$ from Hare [1][2], improving on Iannucci and Sorli [3]
- $t = \omega(N) \geqslant 8$ from Hagis [18] and Chein [19], improving on Pomerance [21]
- $t = \omega(N) \geqslant 11$ if $p_t \geqslant 5$ from both Hagis [14] and Kishore [15]
- $t = \omega(N) \geqslant 17$ if $p_t \geqslant 7$, improving on Norton [25]
- $t = \omega(N) \geqslant 29$ if $p_t \geqslant 11$, improving on Norton [25]
- $p_7 = 5$ if $t = \omega(N) = 8$ from Voight [6]

Suryanarayana and Hagis [23] showed that in all cases $0.596 < \sum_{p|N} \frac{1}{p} < 0.694$. Their paper has more precise bounds based on the divisibility of N by 3 and 5. Cohen [17] gives stricter ranges for the same sum, as well as an argument that such bounds are unlikely to be improved upon significantly.

B Using Sigma Chains

Sigma chains are an easily automated system for proving facts about odd perfect numbers. Each line of the proof starts with a factor known or assumed to divide N, along with its exponent. Since σ is multiplicative, knowledge of this power leads to knowledge of other powers. If any step would lead to an impossible situation (see below), that chain of the proof is terminated and the next possibility is considered.

The following is the beginning of a proof that no odd perfect number has a component less than 10^{30}. The complete proof would take many thousands of pages (and has not been completed); it merely serves of an example of how one constructs such proofs.

The sigma chains are terminated (a line fails to be indented further than the preceding line) is if fails in one of the following ways:

xs: The indicated prime appears more times than it is allowed. If the chain assumes that $3^6 \parallel N$ then a chain with 7 or more factors of 3 is terminated.

overabundant: The abundancy of the prime factors already exceeds 2, so regardless of the other factors, N will fail to be perfect.

The factorizations of the largest half-dozen composites are due to the WIMS (WWW Interactive Multipurpose Server) 'Factoris' at wims.unice.fr.

```
3^6 -> 1093
 1093 -> 2 * 547
  547^2 -> 3 * 163 * 613
   163^2 -> 3 * 7 * 19 * 67
    7^2 -> 3 * 19
     19^2 -> 3 * 127
      127^2 -> 3 * 5419
       5419^2 -> 3 * 31 * 313 * 1009
        31^2 -> 3xs * 331
        31^4 -> 5 * 11 * 17351 overabundant
        31^6 -> 917087137
         917087137^2 -> 841048817767943907 = 3xs * 43 * ...
         917087137^4 -> 707363113097541065394066657400343621 = 31747594185191 * 22280841470107301759731
          31747594185191^2 -> 1007909736547605142797891673 = 2671 * ...
           2671^2 -> 3xs * 7 * 19 * 31 * 577
           2671^4 -> 5 * 11^2 * 571 * 147389551 overabundant
           2671^6 -> 127 * 2860238405785894351
            2860238405785894351^2 -> 8180963737932634435623130240933711201 = 3xs * ...
            2860238405785894351^4 -> 66928167681368702165290385989523603378180978807961717840411453086872862401
            = 5 * 11 * 27362961781 * ... overabundant
            2860238405785894351^6 -> 547536912847552236833797552678177439625986620...
            ...52078327630968478229903427181602178486675491022832132392674545360 1
            = 7xs * 2339 * 337498477 * 101341228212827 * ...
          31747594185191^4 -> 1015882037027398808700619554107312555810842320401403361
          = 5 * 11 * 27581 * ... overabundant
          31747594185191^6 -> 1023917396303697734185706186389856013546714071491865519918239687882024981130882641
          = 29 * 68279 * 17581747 * ...
           29^2 -> 13 * 67
            13^2 -> 3xs * 61
            13^4 -> 30941
             30941^2 -> 157 * 433 * 14083
              157^2 -> 3xs * 8269
              157^4 -> 11 * 31 * 1793161
               1793161^2 -> 3 * 53401 * 20070961
               1793161^4 -> 10338972519025428945627605 = 5 * ... overabundant
               1793161^6 -> 33244204896240857141380146321496272367 = 1163 * ...
                1163^2 -> 1353733
                 1353733^2 -> 1832594389023 = 3xs * 31 * ...
                 1353733^4 -> 3358399713833250034586381
                  3358399713833250034586381^2 -> 7 * 13 * 43 * 4717303 * ...
                   43^2 -> 3xs * 631
                   43^4 -> 3500201
                    3500201^2 -> 13 * 139 * 28411 * 238639
                     139^2 -> 3xs * 13 * 499
                     139^4 -> 41 * 9170881
                      41^2 -> 1723
                       1723^2 -> 3xs * 990151
                       1723^4 -> 6101 * 1445413861

                  3358399713833250034586381^4 ->
                  3358399713833250034586381^6 ->

                 1353733^6 -> 61545799252873846820943201864531528 43 = 113 * 6343 * ...
                  113^2 -> 13 * 991
                   991^2 -> 3xs * 7 * 13^2 * 277
                   991^4 -> 965457315905 = 5 * ... overabundant
                   991^6 -> 948157286261299297 = ??


         917087137^6 -> 594926909354625698481013708377649908321489049770907687 = 7 * 13441 * ...
          13441^2 -> 3xs * 37 * 1627693
          13441^4 -> 32640637833350405 = 5 * ... overabundant
          13441^6 -> 589687333111988200885 8247 = 7xs * 211 * ...
       5419^4 -> 862495334190761 = 1031 * ...
        1031^2 -> 7 * 97 * 1567
         97^2 -> 3xs * 3169
         97^4 -> 11 * 31 * 262321
          11^2 -> 7 * 19xs
          11^4 -> 5 * 3221 overabundant
          11^6 -> 43 * 45319
           43^2 -> 3xs * 631
           43^4 -> 3500201
            3500201^2 -> 12251410540603 = 13 * 139 * ...
             13^2 -> 3xs * 61
             13^4 -> 30941
              30941^2 -> 157 * 433 * 14083
               157^2 -> 3xs * 8269
```

```
157^4 -> 11 * 31 * 1793161
 31^2 -> 3xs * 331
 31^4 -> 5 * 11 * 17351 overabundant
 31^6 -> 917087137
  917087137^2 -> 841048817767943907 = 3xs * 43 * ...
  917087137^4 -> 707363113097541065394066657400343621 = ??
  917087137^6 -> 594926909354625698481013708377649908321489049770907687 = 7 * 13441 * ...
   13441^2 -> 3xs * 37 * 1627693
   13441^4 -> 32640637833350405 = 5 * ... overabundant
   13441^6 -> 5896873331119882008858247 = 7xs * 211 * ...
 1793161^2 -> 3 * 53401 * 20070961
 1793161^4 -> 10338972519025428945627605 = 5 * ... overabundant
 1793161^6 -> 33244204896240857141380146321496272367 = 1163 * ...
  1163^2 -> 1353733
   1353733^2 -> 1832594389023 = 3xs * 31 * ...
   1353733^4 -> 3358399713833250034586381 = ??
   1353733^6 -> 6154579925287384682094320186453152843 = 113 * 6343 * ...
    113^2 -> 13 * 991
     991^2 -> 3xs * 7 * 13^2 * 277
     991^4 -> 965457315905 = 5 * ... overabundant
     991^6 -> 948157286261299297 = ??
```

C Bounds on the Prime Factors of Odd Perfect Numbers

These are the best bounds I have been able to prove, using the various proofs and methods described in the paper.

$$\omega(N) = 8$$

$$10000019 \leqslant p_1 < 10^{19685}$$
$$10007 \leqslant p_2 < 10^{6536}$$
$$101 \leqslant p_3 \leqslant 12884901877$$
$$29 \leqslant p_4 \leqslant 262139$$
$$17 \leqslant p_5 \leqslant 1279$$
$$11 \leqslant p_6 \leqslant 47$$
$$5 \leqslant p_7 \leqslant 5$$
$$3 \leqslant p_8 \leqslant 3$$

$$\omega(N) = 9$$

$$10000019 \leqslant p_1 < 10^{78914}$$
$$10007 \leqslant p_2 < 10^{26305}$$
$$101 \leqslant p_3 < 10^{15783}$$
$$29 \leqslant p_4 \leqslant 17179869143$$
$$19 \leqslant p_5 \leqslant 327673$$
$$13 \leqslant p_6 \leqslant 1531$$
$$11 \leqslant p_7 \leqslant 67$$
$$5 \leqslant p_8 \leqslant 11$$
$$3 \leqslant p_9 \leqslant 3$$

$$\omega(N) = 10$$

$$10000019 \leqslant p_1 < 10^{315653}$$
$$10007 \leqslant p_2 < 10^{105218}$$
$$101 \leqslant p_3 < 10^{63131}$$
$$31 \leqslant p_4 < 10^{45094}$$
$$29 \leqslant p_5 \leqslant 21474836479$$
$$19 \leqslant p_6 \leqslant 393209$$
$$13 \leqslant p_7 \leqslant 1789$$
$$11 \leqslant p_8 \leqslant 79$$
$$5 \leqslant p_9 \leqslant 13$$
$$3 \leqslant p_{10} \leqslant 3$$

$$\omega(N) = 11$$
$$10000019 \leqslant p_1 < 10^{1262612}$$
$$10007 \leqslant p_2 < 10^{420871}$$
$$101 \leqslant p_3 < 10^{252523}$$
$$31 \leqslant p_4 < 10^{180374}$$
$$23 \leqslant p_5 < 10^{140291}$$
$$19 \leqslant p_6 \leqslant 25769803751$$
$$17 \leqslant p_7 \leqslant 458747$$
$$13 \leqslant p_8 \leqslant 2041$$
$$11 \leqslant p_9 \leqslant 97$$
$$5 \leqslant p_{10} \leqslant 17$$
$$3 \leqslant p_{11} \leqslant 5$$

$$\omega(N) = 12$$
$$10000019 \leqslant p_1 < 10^{5050446}$$
$$10007 \leqslant p_2 < 10^{1683482}$$
$$101 \leqslant p_3 < 10^{1010090}$$
$$37 \leqslant p_4 < 10^{721493}$$
$$31 \leqslant p_5 < 10^{561161}$$
$$23 \leqslant p_6 < 10^{459132}$$
$$19 \leqslant p_7 \leqslant 30064771027$$
$$17 \leqslant p_8 \leqslant 524287$$
$$13 \leqslant p_9 \leqslant 2297$$
$$11 \leqslant p_{10} \leqslant 103$$
$$5 \leqslant p_{11} \leqslant 17$$
$$3 \leqslant p_{12} \leqslant 5$$

$$\omega(N) = 13$$

$$10000019 \leqslant p_1 < 10^{20201782}$$
$$10007 \leqslant p_2 < 10^{6733928}$$
$$101 \leqslant p_3 < 10^{4040357}$$
$$41 \leqslant p_4 < 10^{2885969}$$
$$37 \leqslant p_5 < 10^{2244643}$$
$$31 \leqslant p_6 < 10^{1836526}$$
$$23 \leqslant p_7 < 10^{1553984}$$
$$19 \leqslant p_8 \leqslant 34359738337$$
$$17 \leqslant p_9 \leqslant 589811$$
$$13 \leqslant p_{10} \leqslant 2557$$
$$11 \leqslant p_{11} \leqslant 113$$
$$5 \leqslant p_{12} \leqslant 19$$
$$3 \leqslant p_{13} \leqslant 5$$

$$\omega(N) = 14$$

$$10000019 \leqslant p_1 < 10^{80807125}$$
$$10007 \leqslant p_2 < 10^{26935709}$$
$$101 \leqslant p_3 < 10^{16161425}$$
$$43 \leqslant p_4 < 10^{11543875}$$
$$41 \leqslant p_5 < 10^{8978570}$$
$$37 \leqslant p_6 < 10^{7346103}$$
$$31 \leqslant p_7 < 10^{6215933}$$
$$23 \leqslant p_8 < 10^{5387142}$$
$$19 \leqslant p_9 \leqslant 38654705663$$
$$17 \leqslant p_{10} \leqslant 655359$$
$$13 \leqslant p_{11} \leqslant 2815$$
$$11 \leqslant p_{12} \leqslant 131$$
$$5 \leqslant p_{13} \leqslant 19$$
$$3 \leqslant p_{14} \leqslant 5$$

$$\omega(N) = 15$$

$$10000019 \leqslant p_1 < 10^{323228497}$$
$$10007 \leqslant p_2 < 10^{107742833}$$
$$101 \leqslant p_3 < 10^{64645700}$$
$$47 \leqslant p_4 < 10^{46175500}$$
$$43 \leqslant p_5 < 10^{35914278}$$
$$41 \leqslant p_6 < 10^{29384409}$$
$$37 \leqslant p_7 < 10^{24863731}$$
$$31 \leqslant p_8 < 10^{21548567}$$
$$23 \leqslant p_9 < 10^{19013441}$$
$$19 \leqslant p_{10} \leqslant 42949672949$$
$$17 \leqslant p_{11} \leqslant 720887$$
$$13 \leqslant p_{12} \leqslant 3067$$
$$11 \leqslant p_{13} \leqslant 139$$
$$5 \leqslant p_{14} \leqslant 19$$
$$3 \leqslant p_{15} \leqslant 5$$

$$\omega(N) = 16$$

$$10000019 \leqslant p_1 < 10^{1292913987}$$
$$10007 \leqslant p_2 < 10^{430971329}$$
$$101 \leqslant p_3 < 10^{25858279}$$
$$59 \leqslant p_4 < 10^{184701999}$$
$$47 \leqslant p_5 < 10^{143657110}$$
$$43 \leqslant p_6 < 10^{117537636}$$
$$41 \leqslant p_7 < 10^{99454923}$$
$$37 \leqslant p_8 < 10^{86194266}$$
$$31 \leqslant p_9 < 10^{76053764}$$
$$23 \leqslant p_{10} < 10^{68048105}$$
$$19 \leqslant p_{11} \leqslant 47244640255$$
$$17 \leqslant p_{12} \leqslant 786431$$
$$13 \leqslant p_{13} \leqslant 3327$$
$$11 \leqslant p_{14} \leqslant 157$$
$$5 \leqslant p_{15} \leqslant 23$$
$$3 \leqslant p_{16} \leqslant 5$$

$$\omega(N) = 17$$

$$10000019 \leqslant p_1 < 10^{5171655946}$$
$$10007 \leqslant p_2 < 10^{1723885316}$$
$$101 \leqslant p_3 < 10^{1034331190}$$
$$61 \leqslant p_4 < 10^{738807993}$$
$$59 \leqslant p_5 < 10^{574628439}$$
$$47 \leqslant p_6 < 10^{470150541}$$
$$43 \leqslant p_7 < 10^{397819689}$$
$$41 \leqslant p_8 < 10^{344777064}$$
$$37 \leqslant p_9 < 10^{304215056}$$
$$31 \leqslant p_{10} < 10^{272192419}$$
$$23 \leqslant p_{11} < 10^{246269331}$$
$$19 \leqslant p_{12} \leqslant 51539607551$$
$$17 \leqslant p_{13} \leqslant 851967$$
$$13 \leqslant p_{14} \leqslant 3583$$
$$11 \leqslant p_{15} \leqslant 167$$
$$5 \leqslant p_{16} \leqslant 23$$
$$3 \leqslant p_{17} \leqslant 7$$

$$\omega(N) = 18$$

$$10000019 \leqslant p_1 < 10^{20686623784}$$
$$10007 \leqslant p_2 < 10^{6895541262}$$
$$101 \leqslant p_3 < 10^{4137324757}$$
$$67 \leqslant p_4 < 10^{2955231970}$$
$$61 \leqslant p_5 < 10^{2298513754}$$
$$59 \leqslant p_6 < 10^{1880602163}$$
$$47 \leqslant p_7 < 10^{1591278753}$$
$$43 \leqslant p_8 < 10^{1379108253}$$
$$41 \leqslant p_9 < 10^{1216860223}$$
$$37 \leqslant p_{10} < 10^{1088769673}$$
$$31 \leqslant p_{11} < 10^{985077324}$$
$$23 \leqslant p_{12} < 10^{899418426}$$
$$19 \leqslant p_{13} \leqslant 55834574847$$
$$17 \leqslant p_{14} \leqslant 917503$$
$$13 \leqslant p_{15} \leqslant 3839$$
$$11 \leqslant p_{16} \leqslant 179$$
$$5 \leqslant p_{17} \leqslant 29$$
$$3 \leqslant p_{18} \leqslant 7$$

$$\omega(N) = 19$$

$$\begin{aligned}
10000019 \leqslant p_1 &< 10^{82746495136}\\
10007 \leqslant p_2 &< 10^{27582165046}\\
101 \leqslant p_3 &< 10^{16549299028}\\
71 \leqslant p_4 &< 10^{11820927877}\\
67 \leqslant p_5 &< 10^{9194055016}\\
61 \leqslant p_6 &< 10^{7522408649}\\
59 \leqslant p_7 &< 10^{6365115011}\\
47 \leqslant p_8 &< 10^{5516433010}\\
43 \leqslant p_9 &< 10^{4867440891}\\
41 \leqslant p_{10} &< 10^{4355078692}\\
37 \leqslant p_{11} &< 10^{3940309293}\\
31 \leqslant p_{12} &< 10^{3597673702}\\
23 \leqslant p_{13} &< 10^{3309859806}\\
19 \leqslant p_{14} &\leqslant 60129542143\\
17 \leqslant p_{15} &\leqslant 983039\\
13 \leqslant p_{16} &\leqslant 4095\\
11 \leqslant p_{17} &\leqslant 191\\
5 \leqslant p_{18} &\leqslant 29\\
3 \leqslant p_{19} &\leqslant 7
\end{aligned}$$

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